Modularity of Termination of TRS under Fair Strategies *

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Abstract. We define a new notion of fairness for term rewriting system (TRS). We prove the modularity of termination of TRS under such fair strategies, that is, two TRS terminate under fair strategies if and only if their disjoint union terminates under fair strategies. In order to do so, we demonstrate that termination under fair strategies of a TRS is equivalent to the TRS being weakly terminating and globally finite. We then use modularity of weakly terminating and globally finite TRS to obtain our modularity result. We also introduce a notion of probabilistic fairness and show that termination of a TRS under a "reasonable randomized fair strategy" is equivalent to termination of the same TRS under fair strategies. The randomized version of all our results follows straightforwardly.

1 Introduction

Modular verification is mandatory for complex systems, since it decomposes verification into smaller problems. While there is an important body of results for the verification of safety properties, the situation is more complex for liveness propertie. In general fairness conditions are needed to guarantee liveness properties. This is in particular the case of termination, as mentioned in [KZ05, GZ03]. A fairness condition is said to guarantee modular termination if the parallel composition of terminating processes is also terminating under this fairness condition. In this paper, we introduce a new notion of fairness which guarantees modular termination. In contrast with other approaches in the field of TRS, our notion of fairness is based on positions, substitutions and redexes of terms of left part of rules that are visited infinitely often. Another key motivation of this work is to introduce a sufficient notion of fairness that forces termination of the direct sum of two terminating TRS, even if there exist infinite loops (as is the case in the example of Toyama which will be our running example).

Contribution:

– We first introduce a new notion of fair strategy as a way to select fairly the application of a rewriting rule. A fair strategy is a strategy that selects

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infinitely often all successors of a term, provided this term is visited infinitely often.

- We propose a characterization of TRS terminating under fair strategies, based on the fact that a fair strategy terminates if and only if the TRS is *weakly terminating*, meaning that for any term there exists a finite derivation reaching a terminal term, and *globally finite*, also called *quasi-terminating*, meaning that the number of visited terms in any derivation is finite. Using this characterization we deduce that termination under fair strategies is not decidable.
- We prove the modularity of termination of TRS under fair strategies, since weak termination of TRS and being globally finite for a TRS are both modular properties.
- We study the termination of TRS under probabilistic fair strategies. We have identified a subclass of those strategies under which a TRS terminates with probability one if and only if it terminates under fair strategies. We show that these strategies can be defined and implemented, because they only need a bounded amount of memory, or no memory at all. We call those strategies: *reasonable randomized fair strategies*. This result is of importance because if one wants to implement a fair strategy in a deterministic way, then one would need to have an arbitrarily huge memory in order to store the history of the set of previously visited terms.
- We also prove that using any *reasonable randomized fair strategies* is an efficient way to reach a terminal term. We show indeed that termination occurs within a finite mean number of rewriting steps, which is a grant of efficiency.

Related Work: Modularity of termination of TRS is widely studied. In [Toy87] Toyama shows that termination of TRS is not a modular property even if two term rewriting systems do not share any symbol. However in [Rus87], Rusinowitch gives sufficient conditions for the modularity of termination of the disjoint union of TRS. If there is no collapsing rule in one of the TRS and no duplicating rule in the other one, then the disjoint union of those two terminating TRS is terminating. Gramlich gives sufficient conditions for ensuring termination of the union of two TRS possibly sharing constructors [Gra94]. In this context, the projective TRS π is formed of the two rules $\{G(x, y) \to x, G(x, y) \to y\}$ where G is a fresh function symbol. Gramlich defines a $\mathcal{C}_\epsilon$-termination, meaning that a TRS R is $\mathcal{C}_\epsilon$-terminating if $R \cup \pi$ is terminating. This notion is a modular property for the disjoint union of TRS [Gra94, Ohl95], as well as for the union of composable TRS as it is proved by Kurihara and Ohushi in [KO95]. Gramlich also shows in [Gra94] that all TRS whose termination can be shown using a simplification order are $\mathcal{C}_\epsilon$-terminating, whereas the converse is false. This makes the $\mathcal{C}_\epsilon$-termination a useful notion because it is modular and more powerful than techniques using simplification orders to prove termination of TRS. Arts and Giesl introduce dependency pairs in [AG00] for proving termination of TRS, and exploit the modular structure of the dependency pair graph for proving termination of TRS in a modular way in [AG98]. Theoretically speaking, this method is both sound and complete for proving termination of TRS, meaning that for any terminating TRS, there exists a dependency pair graph, and a quasi-order, that prove that the TRS is terminating. However, people who try to prove termination automatically are more often looking for heuristics that can compute the "smallest" approximation of the dependency pair graph – computing the DP graph is undecidable – to minimize the chance of failing to prove termination of a TRS. Urbain in [Urb01] uses Arts and Giesl's work to define a method for incrementally proving termination on TRS defined hierarchically. This method allows him to prove that a TRS is terminating in a modular way. In this work, Urbain considers TRS which can be represented as a set of embedded modules, and incrementally proves termination on modules in a modular way. This technique is implemented in the CiME tool [CM96]. Thiemann, Giesl and Schneider-Kamp improve Urbain's work and implement it in the AProVE termination prover [TGSK04]. Weak termination as well as weak innermost termination of term rewriting are proved to be modular properties in [KK90, BKM89, Dro89]. Gramlich later proves that innermost termination is a modular property of TRS [Gra92].

The reader may notice that none of these existing results encompasses our work on termination of fair strategies. It is not difficult to construct for each method a TRS which terminates under fair strategies but does not fulfill hypotheses required by these different frameworks (most of the time Toyama's example is sufficient). We mention related work about fairness in Section 7, where we show that our notion of fairness is different to the existing notions of fairness.

Outline In the next section, we recall basic rewriting definitions and results that are used in this paper. In Section 3 we introduce the notion of termination under fair strategies. In Section 4, we characterize the set of TRS that terminate under fair strategies and deduce that termination under fair strategies is not decidable. We prove in Section 5 that this termination under fair strategies is modular. In Section 6 we give the definition of randomized fair strategies, and reasonable randomized fair strategies. We prove that a TRS terminates positively and almost surely under reasonable randomized fair strategies if and only if it terminates under fair strategies. In Section 7, we compare our notion with existing notions of fair termination. In the last section we conclude, and describe our perspectives. All usual rewriting notions used in this paper are recalled in Appendix A and all proofs of our results are presented in Appendix B and C for the convenience of the reviewers.

2 Some Notations and Basic Results

In this paper we use the same notations as in [BN98]. In this section we present the main basic definitions and results needed for the rest of the paper. In order to define the notion of derivation, we use the symbol $\perp$ which represents a fictive successor to any terminal state. Notice that the symbol $\perp$ does not belong to the signature of the rewriting system. The only way to reach $\perp$ is to reach a term without any successor. Moreover once $\perp$ is reached no more rule can be applied.

Definition 1 (Derivation). *Given a TRS R over the set of terms $T(\Sigma,V)$, a* derivation *π is an infinite sequence of elements $(\pi_i)_{i\in\mathbb{N}} \in T(\Sigma,V)\cup\{\bot\}$ such that:*

$$(\pi_n)_{n\in\mathbb{N}} \stackrel{\mathsf{def}}{=} \begin{cases} \pi_0 \in T(\Sigma,V) & \\ \forall i \in \mathbb{N}, \pi_i \rightarrow_R \pi_{i+1} & \\ \pi_i = \bot & \Rightarrow \forall j > i, \pi_j = \bot \\ \pi_i \rightarrow_R \bot & \Leftrightarrow \forall t \in T(\Sigma,V), \neg(\pi_i \rightarrow_R t) \end{cases}$$

The third item means that once $\bot$ has been reached, the system artificially loops on this symbol. The last item ensures that if π_i can be rewritten into $\bot$, then it cannot be rewritten into anything else. We note by $\overline{\pi_i}$ the i first elements of π that belong to $T(\Sigma,V)\cup\{\bot\}$.

Definition 2 (Terminal Term, Terminating TRS, Weak Termination). *Given a TRS R over $T(\Sigma,V)$:*

- *A term $t \in T(\Sigma,V)$ is* terminal *if for any term $t' \in T(\Sigma,V)$ $\neg(t \rightarrow_R t')$.*
- *A derivation π is said to be* terminating *if there exists an index i such that $\pi_i = \bot$.*
- *R is terminating if, for all $t \in T(\Sigma,V)$, there exists no infinite derivation $t \rightarrow_R t_1 \rightarrow_R \ldots \rightarrow_R t_n \rightarrow_R \ldots$.*
- *R is said to be* weakly terminating *if for all terms $t \in T(\Sigma,V)$, there exists a terminal term t' such that $t \rightarrow_R^* t'$.*

Example 1 (Toyama [Toy87]). Consider the two following TRS:

$$R_1 = \begin{cases} g(x,y) \rightarrow x \\ g(x,y) \rightarrow y \end{cases}$$
$$R_2 = \left\{ f(0,1,z) \rightarrow f(z,z,z) \right.$$

The systems are disjoint and both terminating, however the disjoint union of TRS $R_1 \uplus R_2$ is not terminating. The following derivation constitutes an infinite loop:

$$f(0,1,g(0,1)) \rightarrow f(g(0,1),g(0,1),g(0,1)) \rightarrow f(0,1,g(0,1))$$

We are interested in finding whether a derivation visits infinitely many different terms or not and if the set of the terms visited infinitely often by a derivation is empty.

Definition 3. *Given a derivation $(\pi_i)_{i\in\mathbb{N}}$, we note by*

$$inf(\pi) \stackrel{\mathsf{def}}{=} \{t \in T(\Sigma,V)\cup\{\bot\} \mid \forall i \in \mathbb{N}, \exists j > i, \pi_j = t\}$$

the set of the terms which are visited infinitely often by π. We also note by $Vis(\pi)$ the set of terms visited by π:

$$Vis(\pi) \stackrel{\mathsf{def}}{=} \{t \in T(\Sigma,V) : \exists i \in \mathbb{N}, \pi_i = t\}$$
$$Vis(\overline{\pi_n}) \stackrel{\mathsf{def}}{=} \{t \in T(\Sigma,V) : \exists i, 1 \le i \le n, \pi_i = t\}$$

and the number of different terms that π, resp. $\overline{\pi_n}$, visits is denoted by:

$$\|\pi\| \stackrel{\text{def}}{=} \#Vis(\pi)$$
$$\|\overline{\pi_n}\| \stackrel{\text{def}}{=} \#Vis(\overline{\pi_n})$$

Definition 4. *We define the set of accessible terms from a term $t \in T(\Sigma, V)$ w.r.t the rewriting relation induced by R, as:*

$$T_s^*(t) \stackrel{\text{def}}{=} \{t' | t \rightarrow_R^* t'\},$$

and the size of this set corresponds to the number of accessible terms from a term t and it is denoted by $\|t\| \stackrel{\text{def}}{=} \#T_s^*(t)$.

We use the same notation for the number of successors of a term and for the number of different terms visited by a derivation. In the rest of the paper the distinction is always clear according to the context. The reader can also immediately notice that:

- $\lim_{n\rightarrow\infty} Vis(\pi_n) = Vis(\pi) \subsetneq T_s^*(\pi_0)$
- $\forall i < j,\ Vis(\pi_i) \subseteq Vis(\pi_j)$
- $inf(\pi) \subseteq Vis(\pi)$ if $inf(\pi) \neq \{\perp\}$
- $inf(\pi) = \perp \Leftrightarrow \pi$ is a terminal derivation
- $inf(\pi) = \{\emptyset\} \Rightarrow \pi$ is a derivation that visits infinitely many different terms.

The converse of the last point is false as shown in Example 2.

Example 2. Consider the usual Peano integer rewriting system which contains the constant 0, the unary function *succ* and the two following rewriting rules:

1. $x \rightarrow succ(x)$
2. $succ(x) \rightarrow x$

We construct a derivation starting from $succ(0)$ which visits all successors of this term with the following constraint: before visiting a new successor the derivation goes back to $succ(0)$. This derivation visits infinitely many different terms by construction, namely all successors of $succ(0)$. Thus π is a derivation visiting infinitely many different terms and such that $inf(\pi) \neq \emptyset$.

Definition 5. *A TRS R over a set of term $T(\Sigma, V)$ is* globally finite *if for all $t \in T(\Sigma, V), \|t\| < \infty$. This is also called quasi-termination according to [Der87].*

Remark 1. A TRS that is terminating is globally finite.

Proposition 1. *Let R be a finite TRS over $T(\Sigma, V)$. If there exists a term $t \in T(\Sigma, V)$ such that $\|t\| = \infty$, then there exists a derivation π such that $\|\pi\| = \infty$ and for all $i \neq j, \pi_i \neq \pi_j$.*

The proof is detailed in Appendix A. A slightly different version of this proposition can be found in [Der87].

3 Termination under Fair Strategies

The notion of fairness is a qualitative notion, which basically means that if an event *can* happen infinitely often, then it *will* happen infinitely often. Our new notion of fairness is given in the following definition.

Definition 6 (Fair Derivation). *Given a TRS R, a derivation* $(\pi_i)_{i\in\mathbb{N}}$ *is* fair *w.r.t to the rewriting relation* $\to_R$*, if :*

$$\forall t \in inf(\pi), \forall t' : (t \to_R t') \Rightarrow t' \in inf(\pi).$$

In other words, for every term t that is visited infinitely often, each term t' such that $t \to_{\mathcal{R}} t'$ is visited infinitely often.

Remark 2. Every derivation $(\pi_i)_{i\in\mathbb{N}}$ such that $inf(\pi) = \emptyset$ is fair. And every derivation $(\pi_i)_{i\in\mathbb{N}}$ such that $inf(\pi) = \perp$ is also fair by definition.

Intuitively, a strategy is an algorithm that chooses a successor to a term, knowing the set of previously reached terms.

Definition 7 (Strategy). *A* strategy ϕ *is a mapping* ϕ *from the set of derivation prefixes to* $T(\Sigma, V) \cup \{\perp\}$*, such that if* $\overline{\pi_n} = \{\pi_0, \ldots, \pi_n\}$*:*

- $\phi(\overline{\pi_n}) \in \{t | \pi_n \to_R t\}$*, if* π_n *is neither a terminal term nor* $\pi_n \neq \perp$*,*
- $\{\perp\}$*, otherwise.*

A strategy gives the way to apply different rules on a term in order to construct a derivation. We define how to construct a derivation under a strategy.

Definition 8 (Derivation under strategy). *A derivation* π *is built under strategy* ϕ*, if* $\pi_0 \in T(\Sigma, V)$ *and for all* $n \in \mathbb{N}$*,* $\pi_{n+1} = \phi(\overline{\pi_n})$*.*

Definition 9 (Termination under strategy). *Let* R *be a* TRS *over* $T(\Sigma, V)$ *and* ϕ *a strategy,* R *is* terminating under the strategy ϕ *if, for all* $t \in T(\Sigma, V)$*, for all derivations* π *under* ϕ *starting from* $\pi_0 = t$*, there exists an index* i *such that* $\pi_i = \perp$ *(meaning that* $\phi(\overline{\pi_{i-1}}) = \perp$*).*

This is equivalent to the following definition: Let R be a TRS over $T(\Sigma, V)$ and ϕ a strategy, R is terminating under the strategy ϕ if there exists no infinite derivation π under ϕ.

Definition 10 (Fair strategy). *A strategy* ϕ *is called* fair *if for any derivation* π *constructed under* ϕ*,* π *is a fair derivation.*

Remark 3. Notice that if a TRS R is terminating, then it is terminating for any strategy ϕ. However the converse is false, as illustrated in example 1, which presents a TRS that generates a loop, whereas it is terminating under the fair strategies we define in this paper. In this case we force the system to exit the loop with our definition of fair strategies. This example is detailed in Section 7.

4 Characterisation of Termination under Fair Strategies

The following theorem presents a characterisation of the TRS terminating under fair strategies.

Theorem 1. *A TRS R over a set of term $T(\Sigma, V)$ terminates under fair strategies if and only if R is globally finite (or quasi-terminating) and R is weakly terminating.*

Proof. $\Rightarrow$

- Let us notice first that if R terminates under fair strategies then that R is weakly terminating.
- We show if R terminates under fair strategies then R is globally finite (it means, for all $t \in T(\Sigma, V), \|t\| < \infty$), by proving that if there exists $t \in T(\Sigma, V)$ such that $\|t\| = \infty$, then there exists a non terminating fair derivation. We already know, thanks to Proposition 1, that there exists an infinite length derivation $(\pi_i)_{i\in\mathbb{N}}$, with $\forall i \neq j, \pi_i \neq \pi_j$. We deduce that $inf(\pi) = \emptyset$, hence π is fair.

$\Leftarrow$ Proving this implication is equivalent to show its contraposition:

$$\exists \text{ non terminating fair derivation } \Rightarrow \vee \begin{cases} \exists t \in T(\Sigma, V), \|t\| = \infty \\ R \text{ is not weakly terminating} \end{cases}$$

Let us consider that there exists a non terminating fair derivation π. We show with a case distinction that the implication holds:

- $inf(\pi) \neq \emptyset$.
  - $inf(\pi) = \{\bot\}$. This is equivalent to π being a terminating derivation, which is not possible because we assume that π is a non terminating fair derivation.
  - $inf(\pi) \neq \{\bot\}$. We suppose that there exists $t \in inf(\pi)$. Because π is fair, we have by Definition 6 that for every t' such that $t \to_R^* t'$, $t' \in inf(\pi)$. This means that all derivations starting from t never reach a terminal state, in other words that R is not weakly terminating starting from t (there exists $t \in T(\Sigma, V)$ such that for any terminal state $t', \neg(t \to_R^* t')$).
- $inf(\pi) = \emptyset$: we know that every term, that is visited by π (π a non terminal fair derivation), is visited only finitely many times. That is to say that for all integers i, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $\forall\ t \in Vis(\overline{\pi_i}), \pi_n \neq t$. Intuitively, the terms visited by the first i steps of π are never visited again by π after N. Then we define the integer sequence $(N_i)_{i\in\mathbb{N}}$ for all $i \in \mathbb{N}$ by:

$$\begin{aligned} N_0 &\stackrel{\text{def}}{=} \min\{n \in \mathbb{N} | \forall k \geq n, \pi_k \neq \pi_0\} \\ N_{i+1} &\stackrel{\text{def}}{=} \min\{n \in \mathbb{N} | \forall k \geq n, \forall t \in Vis(\overline{\pi_{N_i}}), \pi_k \neq t\} \end{aligned}$$

Using the sequence $(N_i)_{i\in\mathbb{N}}$, we can build the following increasing sequence of sets, for the relation $\subsetneq$:

$$Vis(\overline{\pi_0}) \subsetneq Vis(\overline{\pi_{N_0}}) \subsetneq \ldots Vis(\overline{\pi_{N_i}}) \subsetneq Vis(\overline{\pi_{N_{i+1}}}) \subsetneq \ldots$$

It clearly shows that $\|\pi\| = \infty$, which implies that all terms t in $Vis(\pi)$ (Definition 3) satisfy $\|t\| = \infty$.

Example 2 is not fair terminating but is weakly terminating, (it is always possible to return to 0), and trivially not globally finite (there is an infinite of visited terms). Let us consider the following simple TRS: $h(x) \to f(x)$ and $f(x) \to h(x)$, it is not fair terminating by construction, it is globally finite and not weakly terminating. These simple examples emphasize that both conditions of our characterization are necessary.

Undecidability: Tison proves in [Tis88] that fair termination of ground TRS is decidable, by providing a decision algorithm based on tree automata. In [GKM83] the authors prove that it is undecidable whether a (finite) rewriting system is quasi-terminating, as it is mentionned in [Der87]. Using this result and our characterisation we conclude that termination of TRS under fair strategies is also undecidable.

5 Termination under fair strategies is a modular property

We first recall that weak termination is a modular property of TRS as shown in [KK90, BKM89, Dro89]. We also provide a proof of modularity of the globally finite property. Although this result is apparently simple, we are not aware of any reference where it appears. That is why we propose a proof of this result and also our own proof of modularity of weakly termination in Appendix B.

Lemma 1 (Weakly Termination is a modular property [KK90, Dro89, BKM89]). *Let R_1 be a TRS over $T(\Sigma_1, V)$ and R_2 a TRS over $T(\Sigma_2, V)$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$. The TRS $R_1 \biguplus R_2$ weakly terminates if and only if R_1 weakly terminates and R_2 weakly terminates.*

Lemma 2 (Modularity of globally finite property). *Let R_1 be a TRS over $T(\Sigma_1, V)$ and R_2 a TRS over $T(\Sigma_2, V)$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$. The TRS $R = R_1 \biguplus R_2$ is globally finite if and only if R_1 is globally finite and R_2 is globally finite.*

Proofs of the two previous lemma are in Appendix B.

According to our characterization, the modularity of termination under fair strategies is a consequence of the modularity of the two aforementioned lemmas. We can now derive the main theorem which proves that termination under fair strategies is a modular property .

Theorem 2. *Let R_1 be a TRS over $T(\Sigma_1, V)$ and R_2 a TRS over $T(\Sigma_2, V)$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$. The TRS $R_1 \biguplus R_2$ terminates under fair strategies if and only if R_1 terminates under fair strategies and R_2 terminates under fair strategies.*

Because termination of TRS implies fair termination, an obvious corollary is that the disjoint union of two terminating TRS is terminating under fair strategies. We use this result in Section 7 for proving that Example 1 is terminating under fair strategies.

6 Randomized Fair Strategy

Previous work [BK02, BG05, BG06, Gar07, BH03] considered the case of the rules fired according to randomized strategies. So in this section we first lift our definition to the randomized case in the sense of Luca de Alfaro's notion of probabilistic fairness introduced in [Alf99]. Then we show that the randomized definition is equivalent to non-randomized one. This allows us to obtain all our results in the randomized context.

Definition 11 (Randomized strategy). *A randomized strategy is a function ϕ that maps the set of derivations prefixes $\overline{\pi_n}$ to the set of probabilistic distributions over the set Ω, where:*

- $\Omega = \{t | \pi_n \to_R t\}$ *if π_n is neither a terminal term nor equal to $\perp$,*
- $\Omega = \{\perp\}$ *otherwise.*

Example 3. We consider the TRS R over the Peano's integer defined in Example 2. We define ϕ, the randomized strategy such that:

$$\left.\begin{array}{r} P[\phi(\overline{\pi_n}) = succ(succ(X))] = \frac{1}{n} \\ P[\phi(\overline{\pi_n}) = X] = 1 - \frac{1}{n} \end{array}\right\} \text{ if } \pi_n = succ(X)$$

Definition 12 (R-fair and RR-fair strategy). *A* randomized fair strategy*, or* R-fair strategy*, is a randomized strategy such that for any non terminal derivation prefix $\overline{\pi_n}$, and for any t' such that $\pi_n \to t'$,*

$$P[\phi(\overline{\pi_n}) = t'] > 0.$$

A R-fair strategy ϕ is said to be reasonable *(with finite memory), or* RR-fair*, if there exists $N \in \mathbb{N}$ such that for all $\overline{\pi_n}$ with $n > N$*

$$\phi(\overline{\pi_n}) = \phi(\overline{\pi_n} - \overline{\pi_N})$$

where $\overline{\pi_n} - \overline{\pi_N}$ is the natural substraction of two derivation prefixes, i.e., $\pi_{n-N} \dots \pi_n$.

In [BG05], the authors presented a model of probabilistic rewriting systems where rules express probabilistic choices of successors, and techniques to prove positive almost sure termination. Positive almost sure termination is a qualitative notion, meaning that whatever the way you rewrite your terms, the mean number of rewriting steps required to reach a terminal state is finite.

Definition 13 (Almost sure termination under randomized strategy). *A term rewrite system R almost surely terminates under a randomized strategy ϕ, if for any $t \in T(\Sigma, V)$, all derivations π under ϕ such that $\pi_0 = t$*

$$P[\exists n \in \mathbb{N} \ s.t. \ \pi_n = \bot] = 1.$$

Moreover, if the smallest integer τ such that π_τ satisfies $E[\tau] < \infty$, then TRS is said to be positively almost surely terminating under ϕ, noted +a.s terminating for short.

This notion is particulary strong, and proving it is often challenging, even on small instances of probabilistic TRS.

Theorem 3. *A TRS R terminates under fair strategy if and only if it almost surely terminates under RR-fair strategy.*

This theorem is proved in Appendix C.

The following example shows that there exists randomized strategy ϕ such that a TRS can be fair terminating (which is equivalent to be RR-fair terminating) but not terminating almost surely under ϕ.

Example 4. Let R be a TRS terminating under fair strategies but not terminating (for instance the famous example of Toyama). It means that for all $t \in T(\Sigma, V)$ such that $\|t\| < \infty$, there exists a cycle $t_0^\infty \to \ldots \to t_{N-1}^\infty \to t_0^\infty$. We consider the following fair randomized strategy ϕ defined by:

$$P[\phi(\overline{\pi_k}) = t] = \begin{cases} 1 - e^{-k} & \text{if } \ \pi_k = t^\infty_{k \ \mathtt{mod} \ N} \wedge t = t^\infty_{(k+1 \ \mathtt{mod} \ N)} \\ \frac{e^{-k}}{\#\{t' | t_k^\infty \to_R t'\} - 1} & \text{if } \ \pi_k = t^\infty_{k \ \mathtt{mod} \ N} \wedge \pi_k \to t \wedge t \neq t^\infty_{(k+1) \ \mathtt{mod} \ N} \end{cases}$$

$$P[\phi(\overline{\pi_k}) = t] = \frac{1}{\#\{t' | \pi_k \to_R t'\}} \text{ if } \forall i, \ \pi_k \neq t_i^\infty \wedge \pi_k \to t.$$

$$P[\phi(\overline{\pi_k}) = t] = 0 \text{ if } \neg(\pi_k \to t)$$

The strategy ϕ is fair randomized because given $\overline{\pi_n}$ all successors t of π_n satisfy $P[\phi(\overline{t})] > 0$ by construction. Consider the derivation π under strategy ϕ, such that $\pi_0 = t_0^\infty$. We adapt the proof of Theorem 3 to prove that the probability to stay for ever on the cycle $t_0^\infty \to \ldots t_N^\infty \to t_0^\infty$ satisfies the following equation:

$$P[\forall n, \ \pi_n = t^\infty_{n \ \mathtt{mod} \ N}] \geq (1 - e^{-1})^{\frac{1}{1-e^{-1}}} > 0$$

We also have that $P[\forall n, \ \pi_n = t^\infty_{n \ \mathtt{mod} \ N}] \leq P[\forall n, \ \pi_n \neq \bot]$. Using the fact that $P[\exists n, \ \pi_n = \bot] = 1 - P[\forall n, \ \pi_n \neq \bot]$, we conclude that $P[\exists n, \ \pi_n = \bot] < 1$, meaning that this TRS is not almost surely terminating under the fair strategy ϕ.

We notice that the strategy considered in Example 4 is a strategy with an unbounded memory, meaning that the strategy has always access to all history of the derivation.

Theorem 4. *Let R be a fair terminating TRS. Then R is positively almost surely terminating under all RR-fair strategies.*

This theorem is proved in Appendix C.

Remark 4. Almost surely terminating of a TRS does not imply termination as shown by the following example: consider $\Sigma = \{succ, NIL\}$ and the following TRS over $T(\Sigma, V)$:

$$succ(x) \to succ(succ(x))$$
$$succ(x) \to x$$

This TRS positively almost surely terminates under the randomized strategy ϕ defined as follows:

$$P[\phi(succ(x)) = x] = \frac{2}{3}$$
$$P[\phi(succ(x)) = succ(succ(x))] = \frac{1}{3}$$

However, this TRS is not fair terminating according for instance to the following derivation:

$$succ(NIL) \to succ(succ(NIL)) \to succ(succ(succ(NIL))) \to \ldots$$

Using the previous theorem, we can apply a RR-fair strategy in order to guarantee that a TRS is terminating under fair strategies. Using such RR-fair strategies we ensure that the termination will occur within a finite mean number of rewriting steps. Moreover implementing a RR-fair strategy is easier than implementing a fair strategy, because in this case we do not need to store the set of previously visited term for deciding the next rule to take.

7 Comparison of Different Notions of Fairness

Strong fairness have been introduced by Porat and Francez [PF86a] in the context of TRS. This notion of fairness means that if a rule *can* be applied infinitly often, then it *will* be applied infinitly often. Basically, a derivation is fair, if it is finite, or infinite and every rewrite rule that is enabled infinitely often is taken infinitely often. In this context, a rewrite rule $l \to r$ being enabled in a term t means that t contains a redex of l, and this rule being taken means that it is applied on a redex of l, to compute the next term of the derivation. Porat and Francez also investigate the termination of fair derivations of TRS modulo an equational theory – namely E-fair termination – [PF86b]. They prove that if R_1 and R_2 are two E-fair terminating TRS which satisfy the property called "full-commutation" then $R = R_1 \cup R_2$ is E-fair terminating. Meseguer in [Mes05] and Lucas and Meseguer [LM08] introduce new notions of fairness, in the context of TRS. They study and compare the termination of derivations complying with their definitions of fairness and justice, and implement their methods in `Maude` [CDE+03]. They associate a label to rewrite rules, using a labeling function $\mathcal{L}$. Each term of a computation is associated to the atomic proposition $\texttt{enabled}(\alpha)$ if there exists a rule $l \to r$ that satisfies $\mathcal{L}(l \to r)$ and that the current term contains a redex of l. A term appearing in a derivation is labeled $\texttt{taken}(\alpha)$ if a rule labelled α has been applied to the previous term to reach the current one. They consider two notions:

1. Fairness, which corresponds to the notion of strong fairness, strong fairness means that if a rule is infinitely often enabled, then it is infinitely often taken.
2. Justice, which corresponds to the notion of weak fairness, meaning that if a rule is eventually always enabled, then it is infinitely often taken.

According to these definitions, fairness implies justice. From those two definitions, the authors distinguish the following sub-cases cases:

- 1-label $R_{\mathcal{F}}$-rule fairness and one 1-label $R_{\mathcal{F}}$-rule justice.
- $R_{\mathcal{F}}$-rule fairness and $R_{\mathcal{F}}$-rule justice.

The difference between 1-label $R_{\mathcal{F}}$-rule and $R_{\mathcal{F}}$-rule is that in the first case all rules of $R_{\mathcal{F}}$ are labelled with one single label (the same for all the rules) and in the second case all rules in $R_{\mathcal{F}}$ have different labels (all labels are pairwise distinct). In [LM08] they provide a clear comparison of existing notions of fairness in TRS and prove that their notion implies the Porat and Francez one. Moreover it is not difficult to see that these approaches are not modular, using for instance the famous example given by Toyama in [Toy87]. In the rest of this section, we show that our notion of termination under fair strategies and the notion introduced in [LM08] and [PF86a] are not comparable.

Definition given in [LM08, PF86a] does not imply our definition: We recall one example of [LM08]:

$$\begin{array}{l} \alpha : a \to f(a) \\ \beta : a \to g(a, b) \\ \gamma : g(a, b) \to c \end{array}$$

This TRS is not globally finite due to the rule α from the term a it is possible an infinite numbers of successors. Hence according to our definition and characterization this TRS is not terminating under fair strategies, but it is according to the definition of Lucas and Meseguer in [LM08], which also implies that is fair terminating according to the definition given in [PF86a].

Our definition does not imply definition of [LM08, PF86a]: We show that Toyama's TRS is terminating under our definition of termination under fair strategies, but does not terminate under the definition of Lucas and Meseguer, neither under the definition of Porat and Francez.

- We label as follow the rules of the Toyama's TRS:

$$\begin{array}{l} \alpha : g(x, y) \to x \\ \beta : g(x, y) \to y \\ \gamma : f(0, 1, z) \to f(z, z, z) \end{array}$$

We consider the following infinite derivation π, which corresponds to the famous Toyama's cycle, defined by:

$$f(0,1,g(0,1)) \xrightarrow{\gamma} f(g(0,1),g(0,1),g(0,1)) \xrightarrow{\alpha} f(0,g(0,1),g(0,1)) \xrightarrow{\beta} f(0,1,g(0,1)) \ldots$$

On all terms in this derivation, it is possible to apply rules α, β and γ. Therefore in all terms of the derivation these three rules are infinitely often enabled. We notice that these three rules are indeed infinitely often taken, because the loop is constituted of infinite number of repetition of the following sequence of rules: γ, α, β. This derivation is thus R-rule fair. Hence we conclude that this TRS is not R-rule fair terminating according to the definition of [LM08]. Using the same example we can easily prove that this TRS is not fair terminating according to the definition given in [PF86a].

- We prove that Toyama's TRS is terminating under fair strategy by three different ways:
  1. Using direct definitions (Definition 9 and Definition 10): we have to show that for all $t \in T(\Sigma, V)$, for all derivations π under fair strategy ϕ, there exists an index i such that $\pi_i = \perp$. The detailed proof is long and quite technical. The idea of the proof is to assume that there exists an infinite derivation π under fair strategies which does not terminate and show a contradiction using an analysis of the rank function.
  2. Using our characterization (Theorem 1): It is not difficult to be convinced that the Toyama' TRS is weakly terminating (it is always possible to apply rules α and β in order to force the termination) and globally finite (using for instance the modularity of quasi-termination mentioned in [Der87]). Using our characterization in Theorem 1, we conclude.
  3. Using our modularity result (Theorem 2): we split the TRS in two disjoint TRS $R_1 = \{\alpha, \beta\}$ and $R_2 = \{\gamma\}$. These two TRS are terminating by construction, thus they are terminating under fair strategies. Using our modularity result in Theorem 2 we conclude.

Remark 5. The previous example clearly shows that our characterization and result of modularity are easier to apply than the direct definition of termination under fair strategies.

As shown with these two counter-examples, our definition and the definition of Lucas and Meseguer and the one of Porat and Francez are different. The main difference between these notions is that we are looking at the terms produced by a rule but they are looking at the application of a rule. Indeed the granularity of our notion of fairness allows us to prove the termination under fair strategies of the famous Toyama's example. But as seen on Lucas and Meseguer's example, our definition of fairness focus too much on the terms and it is not possible to prove termination in this case. However, in the case where the considered TRS is globally finite, the notion of termination under fairness given by Lucas and Meseguer implies our definition, because it implies weakly termination.

8 Conclusion

We have introduced a new notion of fairness which is in some sense more general than the existing ones as we have shown in the previous section. Moreover it is modular whereas the others are not. We give a surprisingly simple characterization of our notion of fairness, and propose an equivalent randomized version of our termination under fair strategies. We obtain results that are linked to the positive almost sure termination of probabilistic TRS [BG05, BG06, Gar07], and we provide a new criterion for proving this property. We conjecture it can be possible to use an inductive approach as it is done by Gnaedig in [Gna07] for improving our results. This opens perspectives in order to obtain "efficient" implantation of such strategies in a tool. Another future work will be to adapt the approach based on the pair dependency graph proposed in [AG00] for proving weakly termination of a TRS.

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A Basics of Rewriting

In this paper we use the same notations as in [BN98]. In this section we recall the basic notions, proofs and results used in this paper.

Definition 14 (Position, Subterm). *Let be Σ a signature, V a set of variables disjoint from Σ, and $s, t \in T(\Sigma, V)$.*

1. *The set $\mathcal{P}os$ of the positions of a term s is a set of strings over the alphabet of positive integers, which is inductively defined as follows :*
   - *If $s = x \in V$, then $\mathcal{P}os(s) \stackrel{\text{def}}{=} \{\epsilon\}$ where ϵ denotes the empty string.*
   - *If $s = f(s_1, \ldots, s_n)$, then*

   $$\mathcal{P}os(s) \stackrel{\text{def}}{=} \{\epsilon\} \cup \bigcup_{i=1}^{n} \{ip | p \in \mathcal{P}os(s_i)\}.$$

   The position ϵ is called the root position *of the term s, and the function or the variable at this position is called the root symbol of s, denoted $root(s)$.*
2. *For $p \in \mathcal{P}os(s)$, the* subterm *of s at the position p, denoted by $s|_p$, is defined by induction on the length of p :*

   $$\begin{aligned} s|_\epsilon &\stackrel{\text{def}}{=} s, \\ f(s_1, \ldots, s_n)|_{iq} &\stackrel{\text{def}}{=} s_i|_q. \end{aligned}$$

3. *For $p \in \mathcal{P}os(s)$, we define the replacement of the subterm of s at position p by the term t, noted $s[t]_p$, by induction on the length of p:*

   $$\begin{aligned} s[t]_\epsilon &\stackrel{\text{def}}{=} t, \\ f(s_1, \ldots, s_n)[t]_{iq} &\stackrel{\text{def}}{=} f(s_1, \ldots, s_i[t]_q, \ldots, s_n). \end{aligned}$$

Example 5 (Tree representation of a term). In [Toy87] Toyama provides a counterexample for termination of the direct sum of TRS, his description is recalled in Example 1. Figure 1 shows the tree representation of this term: $f(g(0, 1), g(0, 1), X)$. Label inside the tree nodes corresponds to function symbols, constant or variables, and labels near the nodes corresponds to the position of the subterm. The subterm of $f(g(0, 1), g(0, 1), X)$ at position 1 is $g(0, 1)$ and the subterm at position 2.1 is the constant 0.

Definition 15 (Substitution). *A* substitution *σ is a function that maps a finite set of variables $\{X_1, \ldots, X_n\} \subseteq V$ to the set of terms $T(\Sigma, V)$. The set of substitutions is noted $\mathcal{S}ub$.*

Definition 16 (Rewrite rule, TRS, Rewriting relation). *Consider a set of term $T(\Sigma, V)$:*

- *A rewrite rule is an element $(l, r) \in T(\Sigma, V)^2$, where l is not a variable and $Var(r) \subseteq Var(l)$. The rule (l, r) is commonly noted $l \to r$.*
- *A term rewrite system (TRS) R, is a set of rewrite rules.*

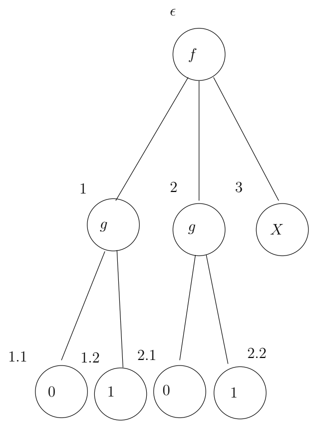

Fig. 1. Tree representation of the term $f(g(0,1), g(0,1), X)$

– *Given a TRS R over a set of term* $T(\Sigma, V)$*, the rewrite relation* $\rightarrow_R \subseteq T(\Sigma, V)^2$ *is a binary relation over the terms, such that :* $t_1 \rightarrow_R t_2$ *if and only if there exists* $p \in \mathcal{P}os(t_1)$*,* $\sigma \in \mathcal{S}ub$*, a rewrite rule* $l \rightarrow r \in R$ *with* $t_1|_p = \sigma(l)$ *and* $t_2 = t_1[\sigma(r)]_p$*.*

Definition 17 (Degree of a term). *The* degree *of a term* t*, denoted by* $deg(t)$*, is the number of terms* t' *such that* $t \rightarrow_R t'$*.*

Definition 18 (Disjoint union of TRS). *Let be* R_1 *and* R_2 *two term rewriting systems over the signatures* Σ_1 *and* Σ_2*. If* Σ_1 *and* Σ_2 *are disjoint, then we note by* $R_1 \uplus R_2$ *the term rewriting system over the set of terms* $T(\Sigma_1 \cup \Sigma_2, V)$ *and call it the disjoint union of* R_0 *and* R_1*.*

Definition 19 (Context). *Let be* $\square$ *a symbol that does not appears in* $\Sigma_k \cup V$*. A* Σ_k-context *is a term of* $T(\Sigma_k, V \cup \{\square\})$*, and can be seen as a term with "holes". Context are denoted by* C*. If* $\{p_1, \ldots, p_n\} = \{p \in \mathcal{P}os(C) | C|_p = \square\}$*, where* p_i *is at the left of* p_{i+1} *in the tree representation of* C*, then*

$$C(t_1, \ldots, t_n) \stackrel{\text{def}}{=} C[t_1]_{p_1} \ldots [t_n]_{p_n}$$

Definition 20 (Pure term). *If there exists* k*, such that* $s \in T(\Sigma_k, V)$*, then* s *is called a* pure *term.*

Given a term s, we write $s = C[s_1, \ldots s_n]$ if $s = C(s_1, \ldots, s_n)$ and

1. $C \neq \square$ is a Σ_k context for some k,
2. $root(s_i) \notin \Sigma_k$ for $i \in \{1, \ldots, n\}$

Definition 21 (Alien terms). *Let be* s *a term such that* $s = C[s_1, \ldots s_n]$*, the* s_i *are called the* alien *subterms of* s*.*

We use the function *rank* which maps the set of the terms $T(\Sigma_1 \uplus \Sigma_2, V)$ where V is a set of variables, two disjoint signatures Σ_1 and Σ_2, to the set of integers.

Definition 22 (Rank of a term). *The* rank *is a function that counts the maximum number of signature changes along all branches of a tree representation of a term. It is defined recursively as below:*

$$rank(t) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } t \text{ is a pure term,} \\ 1 + \max_{i \in \{1 \ldots n\}} \{rank(t_i)\} & \text{if } t = C[t_1, \ldots, t_n] \end{cases}$$

Example 6. Consider the two TRS over the two disjoint signatures $\Sigma_1 = \{g\}$ and $\Sigma_2 = \{f\}$. The rank of the term $f(0, 1, g(0, 1))$ is 2 and the rank of $f(0, 0, 0)$ is one.

Proposition 2. *If a TRS R over $T(\Sigma, V)$ is finite, then for all $t \in T(\Sigma, V)$, $deg(t)$ is finite.*

Proof. Let R be a finite TRS over $T(\Sigma, V)$, we have for all $t \in T(\Sigma, V)$:

$$deg(t) \stackrel{\text{def}}{=} \#\{t \rightarrow_R t'\} \leq \#R \times \#\mathcal{P}os(t),$$

where $\#R$ denotes the the number of rules in R, and $\#\mathcal{P}os(t)$ stands for the number of subterm of t.

Given a term algebra $T(\Sigma, V)$, a finite TRS R, and a term $t \in T(\Sigma, V)$, we define a non deterministic algorithm which computes a covering shortest path tree rooted in t, denoted by $SPT(t)$ (in this notation for simplicity we omit to mention the term algebra and the TRS). Intuitively, $SPT(t)$'s paths correspond to the shortest paths (in number of application of rewriting rules) from t to any term $t' \in T(\Sigma, V)$.

The shortest path tree $SPT(t)$ is composed of a set of nodes, $NSPT(t) \subseteq T(\Sigma, V)$, and a set of edges $ESPT(t)$ are composed of rewriting relations (Definition 16). We describe our algorithm to build $SPT(t)$ in Definition 23. In this construction we use a non-deterministic mapping, denoted by *one*, which chooses an element in a set of elements, defined as follow:

$$one(X) \stackrel{\text{def}}{=} \begin{cases} x \in X, & \text{if } X \neq \emptyset \\ \emptyset, & \text{otherwise.} \end{cases}$$

Definition 23 (Construction of $SPT(t)$). *Let R be a TRS over the term algebra $T(\Sigma, V)$, the set $SPT(t) = (NSPT(t), ESPT(t))$ is the short path tree rooted in t, where $NSPT(t)$ is a set of term, called nodes, and $ESPT(t)$ is a set of edges. We construct $SPT(t)$ inductively by a breadth first search. The shortest path trees of depth n are denoted $SPT_n(t) = (NSPT_n(t), ESPT_n(t))$. By construction $SPT(t)$ is equal to $\bigcup_{n \geq 0} SPT_n(t)$. $SPT_n(t)$ is inductively defined as follow on the number of applications of a rule from the root term t:*

– *Initialization :* $\begin{cases} NSPT_0(t) \stackrel{\text{def}}{=} t \\ ESPT_0(t) \stackrel{\text{def}}{=} \emptyset \end{cases}$
– *Inductive construction:*

$$\begin{cases} NSPT_{n+1}(t) \stackrel{\text{def}}{=} NSPT_n(t) \bigcup_{t' \notin NSPT_n(t) \wedge \exists s \in NSPT_n(t): s \to_R t'} t', \\ ESPT_{n+1}(t) \stackrel{\text{def}}{=} ESPT_n(t) \bigcup_{t' \notin NSPT_n(t)} one(s \to_R t' | s \in NSPT_n(t)) \end{cases}$$

Remark 6. The non-deterministic operator *one* grants that every term has got at most a predecessor. If this operator is removed, then this algorithm computes a directed acyclic graph, with a single maximal element, the term t. To write a deterministic algorithm which complies to this non-deterministic specification, one have to replace the *one* operator by a function that deterministicly chooses one single element of a set. Usual implementations are based upon terms structure, where subterms are visited for example from left to right.

Remark 7. For all $t \in T(\Sigma, V)$, the application $SPT_n(t) : \mathbb{N} \to 2^{T(\Sigma,V)} \times 2^{T(\Sigma,V) \times T(\Sigma,V)}$ is monotonic for the $\subseteq$ relation, i.e.

$$\forall t \in T(\Sigma, V), \forall n \in \mathbb{N} \ \begin{cases} NPST_n(t) \subseteq NPST_{n+1}(t) \\ EPST_n(t) \subseteq EPST_{n+1}(t) \end{cases} \quad (1)$$

In Remark 8 we notice that $SPT(t)$ is a covering tree of the reachable terms from t using the TRS, it means that it captures all possible terms that can be generated from t.

Remark 8. Our algorithm given in Defintion 23 for constructing $SPT_n(t)$ builds:

– the set of nodes $NSPT(t)$ using a breadth first search, the reader can notice that it is just an inductive implantation of the set $T_s^*(t)$.
– the set of edge $ESPT(t)$ using the function one, we construct only one edge to link a new reacheable term to the previous one. This leads to build an tree, it means each node has at most one successor.

Proposition 3. *Let R be a finite term rewrite system over $T(\Sigma, V)$, if there exists a term t with $\|t\| = \infty$ then there exists t' such that $t \to_R^* t'$ with $\|t'\| = \infty$.*

Proof. We make a proof by transposition. Consider a term rewrite system R over $T(\Sigma, V)$ such that for each term t, and for all t' such that $t \to_R t'_i$ there exists $k_i \in \mathbb{N}$ such that $\|t'_i\| \le k_i$. Let's consider a term t, we denote by $\{t'_1, \ldots, t'_{deg(t)}\}$ the set of the successors of t, then we obtain:

$$\|t\| \le \sum_{i=1}^{deg(t)} (1 + \|t'_i\|) \Rightarrow \|t\| \le \sum_{i=1}^{deg(t)} (1 + k_i) = deg(t) \times (1 + k_i)$$
$$\Rightarrow \|t\| \le deg(t) \times [1 + \max_{i \in \{1,\ldots,deg(t)\}} k_i] < \infty$$

We conclude thanks to Proposition 2 ($deg(t)$ is finite) and the fact that R is a finite term rewrite system.

We recall the Koenig's Lemma.

Lemma 3. *A finitely branching tree is infinite iff it contains an infinite length path.*

Lemma 4. *If $s \to t$ in a TRS then $rank(s) \geq rank(t)$.*

Detailed proof is given in [BN98].

We extend Definition 4 in order to consider the terms reachable with a given rank.

Definition 24. *We define the set of accessible terms with the rank i from a term $t \in T(\Sigma, V)$ w.r.t the rewriting relation induced by R, as :*

$$T_s(t,i) \stackrel{\text{def}}{=} \{t' | t \to_R^* t' \text{ such that } rank(t') = i\}$$

We deduce from Lemma 4 the following corollary.

Corollary 1. *Given t a term of $T(\Sigma_1 \cup \Sigma_2, V)$ then*

1. $\forall i > rank(t)$, *then* $T_s(t,i) = \emptyset$
2. $$\|t\| = \sum_{i=0}^{i=rank(t)} \#T_s^*(t,i)$$

Proof. 1. The first point is proved using Lemma 4 and the definition of $T_s(t,i)$. According to Lemma 4 $rank(t) \leq rank(t')$, hence $\forall i > rank(t)$, then we have $T_s(t,i) = \emptyset$.
2. The second point is just a direct consequence of the first point and the definition.

$$\|t\| = \#T_s^*(t) = \#\{t' | t \to_R^* t'\} = \sum_{i=0}^{i=rank(t)} \#T_s^*(t,i)$$

Proposition 1. *Let R be a finite TRS over $T(\Sigma, V)$. If there exists a term $t \in T(\Sigma, V)$ such that $\|t\| = \infty$, then there exists a derivation π such that $\|\pi\| = \infty$ and for all $i \neq j, \pi_i \neq \pi_j$.*

Proof. Let t be a term such that $\|t\| = \infty$. We have shown in Remark 8 that $SPT(t)$ is a tree with the following property $T_s^*(t) = NSPT(t)$. By Koenig's lemma (recall in lemma 3 in appendix), the short path tree $SPT(t)$ contains infinitely many nodes if and only if it contains an infinite path. Therefore, there exists an infinite path $(\pi_i)_{i \in \mathbb{N}}$ in $SPT(t)$, such that $\pi_0 = t$. Because π is a path belonging to a tree, it contains no cycles. By construction of $SPT(t)$ we also have that for all $i \neq j$ $\pi_i \neq \pi_j$ and $\|\pi\| = \infty$.

B Proofs of Modularity

Lemma 1 (Weakly Termination is a modular property [KK90, Dro89, BKM89]). *Let R_1 be a TRS over $T(\Sigma_1, V)$ and R_2 a TRS over $T(\Sigma_2, V)$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$. The TRS $R_1 \biguplus R_2$ weakly terminates if and only if R_1 weakly terminates and R_2 weakly terminates.*

Proof. $\Rightarrow$ This way of the demonstration is immediate, using a kind of projection.

$\Leftarrow$ Let suppose that R_1, resp. R_2 weakly terminates. We demonstrate that for all $t \in T(\Sigma_1 \uplus \Sigma_2, V)$, t is weakly terminating by induction on the rank of t.

- Base case: $rank(t) = 0$, it means that t is a pure term, w.l.o.g we can consider that $t \in T(\Sigma_1, V)$. In this case, by hypothesis we know that there exists a derivation from t that leads to a terminal term. We conclude that the property holds for the base case.
- We suppose that for all terms $t' \in T(\Sigma_1 \uplus \Sigma_2, V)$ such that $rank(t') \leq k$, there exists a derivation from t' that leads to a terminal term. We prove that for all $t = C[s_1, \ldots s_n] \in T(\Sigma_1 \uplus \Sigma_2, V)$ with $rank(t) = k + 1$ there exists a derivation from t that leads to a terminal term.
  We can suppose, w.l.o.g that $root(t) \in \Sigma_1$. We notice that all the alien terms have a rank less or equal than k, by definition of the rank. Hence, using the induction hypothesis, for all these alien terms there exists a derivation leading to a terminal term. This allows us to see all alien terms in t as some constant terms in Σ_1. Using the fact that the rewriting system R_1 is weakly terminating, we conclude that there exists a derivation from t to a terminal term. This concludes the proof.

Lemma 2 (Modularity of globally finite property). *Let R_1 be a TRS over $T(\Sigma_1, V)$ and R_2 a TRS over $T(\Sigma_2, V)$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$. The TRS $R = R_1 \biguplus R_2$ is globally finite if and only if R_1 is globally finite and R_2 is globally finite.*

Proof. $\Rightarrow$ This way of the demonstration is immediate, the idea is to consider only terms in one signature and restrict the application of rules of this signature for the weakly termination.

$\Leftarrow$ Consider t_0 be the first term of an infinite length derivation $D = (t_i)_{\{i \in \mathbb{N}\}}$ in $R = R_1 \biguplus R_2$, we show that it contains finitely many different terms. We demonstrate this result by induction on the rank function.

- Base case: $rank(t_0) = 0$, it means that all elements of the term t_0 are in the signature of R_1 or R_2. By hypothesis these two TRS terminate, thus we know that every derivation D associated to t_0 have a finite number of terms.
- Induction: Assume that for all terms t such that $rank(t) = n$ the result holds, then we want to prove the lemma for t_0 such that $rank(t_0) = n + 1$. We first decompose $\|t_0\|$ using corollary 1:

$$\begin{aligned}\|t_0\| &= \sum_{i=0}^{i=rank(t_0)} \#T_s^*(t_0, i) \\ &= \sum_{i=0}^{i=rank(t_0)-1} \#T_s^*(t_0, i) + \#T_s^*(t_0, rank(t_0))\end{aligned}$$

The first part is finite by induction hypothesis, based on the fact that the rank is at most n for all these terms. In the following, we focus on the second term and recall the definition.

$$\#T_s^*(t_0, rank(t_0)) = \#\{t' | t_0 \rightarrow^* t', rank(t') = rank(t_0)\}$$

According to Lemma 4 on the rank, we deduce that all term in a such derivation have the same rank, otherwise it is impossible to recover the initial rank in a derivation after you passed through a term with a smaller rank. Hence we have only to consider derivation where for all, $\hat{t}$ such that $t_0 \rightarrow^* \hat{t}$, we have $rank(t_0) = rank(\hat{t_0})$.
This implies that there exists a branch a each step of rewriting such that $t_0 = C[s_1, \ldots, s_n]$, and $\exists i \in \{0, \ldots, n\}$ such that $rank(t_0) = 1 + rank(s_i)$. We can conclude making an approximation of the number of possible terms reachable from t_0 with the same rank.

$$\#T_s^*(t_0, rank(t_0)) \leq \|t_0\|_{\rhd} \times T_s^*(s_i)^{max(arity(R_1), arity(R_2))}$$

is finite, where $\|t_0\|_{\rhd}$ is defined as the restriction to outer rewriting terms in the definition of $\|t_0\|$. Formally, if $t_0 = C[s_1, \ldots, s_n]$, $\|t_0\|_{\rhd}$ is defined by $\#\{C'[s'_1, \ldots, s'_m] | C[s_1, \ldots, s_n] \rightarrow^*_{R_1} C'[s'_1, \ldots, s'_m]\}$. By construction the arity of the two rewriting systems are finite. By induction hypothesis for all i, $T_s^*(s_i)$ is finite because these terms have a smaller rank. Finally $\|t_0\|_{\rhd}$ is finite because $t_0 = C[s_1, \ldots, s_n]$ and R_1 is globally finite

C Proofs of Randomized Part

Theorem 3. *A TRS R terminates under fair strategy if and only if it almost surely terminates under RR-fair strategy.*

Proof. $\Rightarrow$ We consider that R is a TRS terminating under fair strategy, according to our characterization in Theorem 1, the both assertions below hold:

(1) $\forall t \in T(\Sigma, V)\ \|t\| < \infty$
(2) $\forall t \in T(\Sigma, V), \exists t'$ s.t. $(\forall t'' \in T(\Sigma, V), \neg(t' \rightarrow_R t''))$.

We prove that $(1) \wedge (2) \Rightarrow (3)$, where

$$(3) \begin{cases} \forall t \in T(\Sigma, V), \forall \phi \text{ RR-fair strategy} \\ \forall (\pi)_{n \in \mathbb{N}} \text{ derivation under } \phi \\ P[\exists n \text{ such that } \pi_n = \bot] = 1 \end{cases}$$

We first show that, once a RR-strategy ϕ and a term $t_0 \in T(\Sigma, V)$ are chosen, then we have the following property for all $t \in T_s^*(\pi_0)$:

$$\forall t' \text{ such that } t \to t', \forall \overline{\pi_n} : \pi_n = t, P[\phi(\overline{\pi_n}) = t'] > Pmin_\phi(t, t') > 0$$

Where

$$Pmin_\phi(t, t') \stackrel{\texttt{def}}{=} \inf_{\overline{\pi_n}, \pi_n = s} \{P[\phi(\overline{\pi_n}) = s']\}.$$

The strategy ϕ is a RR-fair strategy, we have

$$\{\phi(\pi_0 \ldots \pi_n) : \pi_n = t\} = \{\phi(\pi_{n-N} \ldots \pi_n) : \pi_n = t\},$$

Thanks to hypothesis (1), $\|\pi_0\|$ is finite. Hence the number of different derivation prefix $\overline{\pi_n} - \overline{\pi_N}$ is bounded by

$$\#\{\pi_{n-N} \ldots \pi_n : \pi_n = t\} \leq \|\pi_{n-N}\|^N \leq \|\pi_0\|^N$$

Because a randomized strategy is defined as a function we have that:

$$\#\{\phi(\pi_0 \ldots \pi_n) : \pi_n = t\} = \#\{\phi(\pi_{n-N} \ldots \pi_n) : \pi_n = t\}$$

Assume there exists a non terminal term t such that $t \in inf(\pi)$ almost surely (denoted a.s). Under this hypothesis, there exists almost surely infinitely many $n \in \mathbb{N}$ for which $\pi_n = t$. Therefore, in this case, the inf operator can be replaced by a min, since we are looking forward the smallest value among a finite set:

$$Pmin_\phi(t, t') = \min_{\overline{\pi_n}, \pi_n = t} \{P[\phi(\overline{\pi_n}) = (t']\}.$$

Because we are dealing with a fair strategy we have for all $\overline{\pi_n}$ such that $\pi_n = t$:

$$1 \geq P[\phi(\pi_0, \ldots, \pi_n) = t'] > 0$$

Hence

$$1 \geq Pmin_\phi(t, t') > 0$$

Now we can better understand how a derivation evolves and which is the probability that the strategy chooses a successor of the last term of a derivation prefix which is closer to a terminal term. We recursively define the sets pre^n, as follow:

$$\begin{array}{lll} pre^0 & \stackrel{\texttt{def}}{=} & \{t | t \text{ is terminal }\} \\ pre^{n+1} & \stackrel{\texttt{def}}{=} & \left\{ \begin{array}{l} t | \exists t' \in pre^n, t \to t' \text{ such that} \\ \quad \wedge\, t \to t'', t'' \notin pre^i, \forall i \in \{1, \ldots, n-1\} \end{array} \right\} \\ pre^\infty & \stackrel{\texttt{def}}{=} & \left\{_{t | \forall t' \text{ terminal}, \neg(t \to^* t')}\right\} \end{array}$$

One immediate property of this set is that a rewriting rule can only decreases by one when it is changing set, but can increase more than one. Formally we express it by the following property: Let R be a TRS, $t_i \in pre^x$ and $t_{i+1} \in pre^y$ be two terms such that $t_i \rightarrow_R t_{i+1}$ and $x > y$ then we have $y = x - 1$. The proof of this property is an immediate consequence of the definition of pre^n.

The set pre^n, is the set of term t such that the smallest derivation prefix starting from t and that leads to a terminal term has for length n. Because of hypothesis (1), we know that $\|\pi_0\|$ is bounded, and because of hypothesis (2), we know that for all non terminal $\pi_i \in \pi$, $\pi_i \in \bigcup_{k=1}^{k=\|\pi_0\|} pre^k$. The latter equality is justified by the fact that each term π_i is a successor of π_0 and that π_i can be rewritten in a terminal term using a number of rewriting steps less than $\|\pi_0\|$.

Given any $1 \leq k \leq M$, we define $Pmin_\phi(k+1,k) \stackrel{\mathsf{def}}{=} min\{Pmin_\phi(t,t') : t \in pre^{k+1} \wedge t' \in pre^k\}$. the minimal probability for the strategy ϕ to chose a term closer to the set of terminal terms when the last term of the derivation prefix $\overline{\pi_n}$ belongs to pre^{k+1}. Clearly $Pmin_\phi(k+1,k) > 0$, because it is the minimal value chose among a finite set of values, each being strictly positive.

Due to the hypothesis (2) (weak termination) we deduce that for all $k \in \mathbb{N}$, there exists a $n \in \mathbb{N}$ such that $\pi_k \in pre^n$ and $n \leq \|\pi_0\|$. We also deduce from the definition of the sets pre^n, that for every $k \in \mathbb{N}$ such that $\pi_k \in pre^{n+1}$ $P[\pi_{k+1} \in pre^n] \geq Pmin_\phi(n+1,n) > 0$.

For the sake of readability, we will note $P_{M,n}$ the set $P_{M,n} \stackrel{\mathsf{def}}{=} \cup_{j=n}^{M} pre^j$.

We show that the latter equation implies that with probability one, there exists k such that $\pi_k \in pre^0$, i.e. $\pi_{k+1} = \perp$. We note by M the integer defined as $M \stackrel{\mathsf{def}}{=} max\{n : \exists t \in T_s^*(\pi_0) \wedge t \in pre^n\}$.

We prove that the two propositions below are equivalent if R is a fair terminating TRS:

(A) $\forall k \in \mathbb{N}, \pi_k \neq \perp$
(B) $\exists n, M > n \geq 1$ s.t. $\forall k \in \mathbb{N},\ \pi_k \in \bigcup_{i=n}^{M} pre^i$

Proof:

$(B) \Rightarrow (A)$: In this part of the proof, we do not need the fact that R is a fair terminating TRS, it is just by definition of pre^i. According to (B) we deduce that for all $k \in \mathbb{N}, \pi_k \notin pre^0$ which implies that that for all $k \in \mathbb{N}, \pi_k \neq \perp$

$(A) \Rightarrow (B)$: We first notice that we have $\pi_k \in pre^0 \Leftrightarrow \pi_k$ is a terminal term which only implies that $\pi_{k+1} = \perp$. (A) implies that $\pi_k \in \bigcup_{i=1}^{\infty} pre^i$. Using our characterization of fair terminating TRS (Theorem 1), we have that for all $\pi_0 \in T(\Sigma, V), \|\pi_0\| < \infty$, it means that there exists M such that $\|\pi_0\| \leq M$. Then due to the weak termination for all $k \in \mathbb{N}, \pi_k \in T_s^*(\pi_0)$ this implies that $\pi_k \in \bigcup_{i=n}^{M} pre^i$.

Hence we have that $P[A] = P[B]$. We show that the probability of the event (B) is 0. We show that the events (B) is a subset of a set of event B' whose probability is zero. We recall that if $B \subseteq B'$ then $P[B] \leq P[B']$. We denote by (B') the following event:

$$B' \stackrel{\mathsf{def}}{=} \exists n \in \mathbb{N},\ \exists R \in \mathbb{N}, \forall k \geq R,\ \pi_k \in P_{M,n}$$

The event (B') means that a derivation stays forever in a set $P_{M,n}$, after a finite index R. The event B is as well an event B' for which $R = 0$. The converse however is false. We show by induction on n, that the probability that a derivation π stays forever in a set $P_{M,n}$ after a finite index R is 0.

– Base case: we suppose that $n = M$. Here we consider that the probability that there exists R is positive, otherwise the left member of this equation is equal to zero:

$$\begin{aligned}
P[\exists R, \forall k \geq R,\ \pi_k \in P_{M,M}] &= P[\exists R, \pi_R \in P_{M,M} \cap \forall k \geq R,\ \pi_k \in P_{M,M}] \\
&= P[\forall k \geq R,\ \pi_k \in P_{M,M} | \pi_R \in P_{M,M}] \\
&\quad \times P[\exists R, \pi_R \in P_{M,M}] \qquad (2) \\
&= P[\bigcap_{k \geq R} \pi_k \in P_{M,M} | \pi_R \in P_{M,M}] \\
&\quad \times P[\exists R, \pi_R \in P_{M,M}]
\end{aligned}$$

Equation 2 is deduced from the line above using the fact that $P[A \cap B] = P[A|B] \times P[B]$ if $P[B] > 0$. By definition of pre^M we know that if $\pi_k \in P_{M,M}$, then $\pi_{k+1} \in P_{M,M}$ or $\pi_{k+1} = P_{M,M-1}$, because M is the integer such that there exists no integer n greater than M for which a successor of π_0 belongs to pre^n. Because of hypothesis (2), for all terms t, there exists $t' \in P_{M,M-1}$ such that $t \to t'$, however, there exists no t'' such that $t \to_R t''$ and no integer n with $t'' \in P_{M,M-n}$. Therefore, $\forall k \in \mathbb{N}$, the following equation holds:

$$P[\pi_{k+1} \in P_{M,M} | \pi_k \in P_{M,M}] = 1 - P[\pi_{k+1} \in P_{M,M-1} | \pi_k \in P_{M,M}]$$

As well as its obvious consequence:

$$P[\pi_{k+1} \in P_{M,M} | \pi_k \in P_{M,M}] \leq 1 - Pmin_\phi(M, M-1). \qquad (3)$$

Notice that equation 3 holds whatever are the values taken by the random values $\pi_0, \ldots, \pi_{k-1}$. Now, we show that the term $P[\bigcap_{k \geq R} \pi_k \in P_{M,M} | \pi_R \in P_{M,M}]$ in equation 2 is equal to zero:

$$P[\bigcap_{k \geq R} \pi_k \in P_{M,M} | \pi_R \in P_{M,M}] = P[\bigcap_{k+1 > R} \pi_{k+1} \in P_{M,M} | \bigcap_{i=R}^{k} \pi_i \in P_{M,M}]$$

By equation 3, we know that for all $k + 1 > R$

$$P[\pi_{k+1} \in P_{M,M} | \bigcap_{i=R}^{k} \pi_i \in P_{M,M}] \leq 1 - Pmin_\phi(M, M-1),$$

as far as the event $\pi_k \in P_{M,M}$ belongs to the field of events generated by $\bigcap_{i=R}^{k} \pi_k \in P_{M,M}$. We then deduce that

$$\begin{aligned}
P[\bigcap_{k \geq R} \pi_k \in P_{M,M} | \pi_R \in P_{M,M}] &= \prod_{k+1 > R} (1 - Pmin_\phi(M, M-1)) \\
&= 0.
\end{aligned}$$

– Let us show that if there exists an integer $n \geq 1$ such that the property $H(n+1)$ holds, then the property $H(n)$ holds as well, where properties $H(n+1)$ and $H(n)$ are defined below:

$$\begin{array}{ll} H(n+1) & P[\exists R \in \mathbb{N},\ \forall k > R,\ \pi_k \in P_{M,n+1}] = 0 \\ H(n) & P[\exists R' \in \mathbb{N},\ \forall k > R',\ \pi_k \in P_{M,n}] = 0. \end{array}$$

In order to compute the aforementioned probabilities, we need to define the following sequence of random variables:

$$\begin{cases} \tau_{1,n} \stackrel{\texttt{def}}{=} \min\{i \in \mathbb{N} : \pi_i \in pre^n\} \\ \tau_{k+1,n} \stackrel{\texttt{def}}{=} \min\{i \in \mathbb{N} : \pi_i \in pre^n \wedge\ i > \tau_{k,n}\} \end{cases}$$

The sequence of $(\tau_{i,n})_{i\in\mathbb{N}}$ is a sequence of stopping times, and $\tau_{i,n}$ is the index of the derivation π for which $\pi_{\tau_{i,n}}$ is the i^{th} element of π that belongs to pre^n. Let us recall that if the i^{th} term of a derivation π, satisfies $\pi_i \in P_{M,n}$, then the only condition that allows π_{i+1} to be selected in pre^{n-1}, is that π_i belongs to pre^n. In other words there must exists an integer k such that $i = \tau_{n_k}$. For proving, that $H(n+1)$ implies $H(n)$, we first show that $H(n+1)$ implies the property (τ), where:

$$(\tau) \begin{cases} \text{If } \tau_{i,n} < \infty \text{ and } \pi_{\tau_{i,n}+1} \in P_{M,n} \\ \text{then } P[\tau_{i+1,n} < \infty] = 1. \end{cases}$$

We prove that under hypothesis $H(n+1)$, property (τ) holds.

Proof. We consider the two complementary cases below:

- If $\pi_{\tau_{i,n}+1} \in pre^n$, then $\tau_{i+1,n} = \tau_{i,n} + 1$, by definition of sequence $(\tau_{i,n})_{i\in\mathbb{N}}$.
- We consider now the case where $\pi_{\tau_{i,n}+1} \in P_{M,n+1}$.
  $H(n+1) \Leftrightarrow P[\forall R \in \mathbb{N},\ \exists k \geq R\ \pi_k \notin P_{M,n+1}] = 1$ By setting $R = \tau_{i,n} + 1$, we got that almost surely there exists an integer $k > \tau_{i,n}$ such that $\pi_k \notin P_{M,n+1}$ and let be k' the smallest k. This value satifies $\pi_{k'} \in pre^n$, therefore $\tau_{i+1,n} = k'$.

We have shown, that $\tau_{i+1,n}$ almost surely exists finite in the both complementary cases, therefore $\tau_{i+1,n}$ almost surely exists finite. □

We show that the property $H(n+1)$ implies the following implication:

$$(B')\ \exists R \in \mathbb{N},\ \forall k \geq R,\ \pi_k \in P_{M,n} \Rightarrow (B'')\exists R' \in \mathbb{N},\ \forall i \geq R'\ \pi_{\tau_{i,n}+1} \in P_{M,n}$$

The events (B') and (B'') are in fact equivalent, but we do not need to prove the equivalence since we show that $P[B'] \leq P[B''] = 0$.

Proof. To prove that (B'') holds, we have to show that almost surely the stopping time sequence $(\tau_i)_{i\in\mathbb{N}}$ exists and that each τ_i is a finite value, almost surely.
Hypothesis (B'), means there exists an integer R, such that for any integer k greater or equal than R, $\pi_k \in P_{M,n}$. The induction hypothesis $H(n+1)$

is sufficient to prove that the stopping time $\tau_{0,n}$ exists finite almost surely, because $H(n+1)$ means the probability that the sequence π_k stay forever in $P_{M,n+1}$ equals zero. The following equivalence even holds under hypothesis (B'),

$$P[\forall k \geq R,\ \pi_k \in P_{M,n+1}] = 0 \Leftrightarrow P[\exists k \geq R,\ \pi_k \in pre^n] = 1$$

The right hand side of this equivalence is deduced from the left one, because under hypothesis (B'), for all integers k, π_k either belongs to $P_{M,n+1}$ or belongs to pre^n and the two sets $P_{M,n+1}$ and pre^n are disjoint. This means, that almost surely, there exists an integer k such that $\pi_k \in pre^n$ a.s., i.e. there exists an integer R' such that $k = \tau_{R',n}$. Hypothesis (B') entails that $\pi_{\tau_{R',n}+1} \in P_{M,n}$, therefore using the property (τ) we deduce that the sequence of stopping time $(\tau_{i,n})_{i\in\mathbb{N}}$ almost surely exists finite, using property (τ) as an induction step. Let us suppose $\tau_{i,n}$ almost surely exists, then because of (B') $\pi_{\tau_{i,n}+1} \in P_{M,n}$, hence by property (τ), $\tau_{i+1,n}$ almost surely exists finite. At this point, we proved (B'').

□

We now have all the required material to compute the probability of the event (B''). We use the two complementary events:

$$\mathcal{K} \stackrel{\texttt{def}}{=} \tau_{0,n} < \infty$$
$$\overline{\mathcal{K}} \stackrel{\texttt{def}}{=} \tau_{0,n} = \infty$$

$$\begin{aligned} P[B''] &= P[B'' \cap\ \mathcal{K}] + P[B'' \cap\ \overline{\mathcal{K}}] \\ &= P[B''|\ \mathcal{K}] \times P[\mathcal{K}] + P[B''|\ \overline{\mathcal{K}}] \times P[\overline{\mathcal{K}}] \end{aligned}$$

We first show that $P[B''|\ \overline{\mathcal{K}}] = 0$. We consider the two complementary situations:

- If $\pi_0 \notin P_{M,n}$ then for all $k \in \mathbb{N}$ $\pi_k \notin P_{M,n}$. This assertion is true, because in this case, if there exists $\pi_k \in P_{M,n}$ then there exists $\tau_{0,n} \leq k$. Therefore, in this case $P[B''|\ \overline{\mathcal{K}}] = 0$.
- If $\pi_0 \in P_{M,n}$, then the event $P[B''|\ \overline{\mathcal{K}}]$ means that $\pi_0 \in P_{M,n+1}$ (otherwise $\tau_{0,n} = 0$) and that for all $k \in \mathbb{N}, \pi_k \in P_{M,n+1}$. The latter event occurs with propability 0, because of the induction hypothesis $H(n+1)$.

We now have to demonstrate that $P[B''|\ \mathcal{K}] = 0$. Knowing that $\mathcal{K}$ holds, we know that almost surely $\tau_{0,n}$ exists, therefore we can compute $P[B''|\ \mathcal{K}]$, as follow:

$$P[\exists R, \forall k > R,\ \pi_k \in P_{M,n}|\ \mathcal{K}] = P[\bigcap_{k\geq R'} \phi(\overline{\pi_{\tau_{k,n}}}) \in P_{M,n}|\ \mathcal{K}].$$

Because of the following $P[\phi(\overline{\pi_{\tau_{k,n}}}) \in P_{M,n}] \leq 1 - Pmin_\phi(n, n-1)$, we get:

$$\begin{aligned} P[\exists R, \forall k > R,\ \pi_k \in P_{M,n}|\ \mathcal{K}] &\leq \prod_{k\geq R'} (1 - Pmin_\phi(n, n-1,)) \\ &= 0 \end{aligned}$$

Which ends the proof by induction.

We have proved that $P[B''] = 0$, thus $P[B'] = 0 = P[A]$. Hence we conclude that

$$P[\exists k \in \mathbb{N}, \pi_k = \bot] = 1$$

.

Proof of the converse: $\Leftarrow$.
We prove the converse by transposition:

$$\neg(1) \;\; \exists t \in T(\Sigma, V) \; \|t\| = \infty$$
$$\neg(2) \;\; \exists t \in T(\Sigma, V), \forall t \to_R^* t' \neg(\forall t'' \in T(\Sigma, V), \neg(t' \to_R t'')).$$

We prove that $\neg(1) \vee \neg(2)$ entails

$$\neg(3) \begin{cases} \exists t \in T(\Sigma, V), \exists \phi \text{ RR-fair strategy} \\ \exists (\pi)_{n \in \mathbb{N}} \text{ derivation under } \phi \\ \text{s.t. } P[\exists n \text{ s.t } \pi_n = \bot] < 1 \end{cases}$$

The easiest part to show, is $\neg(2) \Rightarrow \neg(3)$. Formula $\neg(2)$ means there exists a term t from which no terminal term is reachable. Therefore, for any RR-fair strategy ϕ and any derivation π under ϕ such that $\pi_0 = t$, $P[\exists n \; \pi_n = \bot] = 0$.

Now, let's prove $\neg(1) \Rightarrow \neg(3)$. Let be $t \in T(\Sigma, V)$ with $\|t\| = \infty$. We proved in proposition 1 the existence of a derivation $(t_i^\infty)_{i \in \mathbb{N}}$ such that $t_0^\infty = t$ and $\forall i \neq j \; t_i^\infty \neq t_j^\infty$. Let us consider ϕ a randomized strategy satifying the following equations:

$$P[\phi(\overline{\pi_k}) = t] \stackrel{\text{def}}{=} \begin{cases} 1 - e^{-i} & \text{if } \; \pi_k = t_i^\infty \wedge t = t_{i+1}^\infty \\ \frac{e^{-i}}{\#\{t_i^\infty \to_R t'\} - 1} & \text{if } \; \pi_k = t_i^\infty \wedge \pi_k \to t \wedge t \neq t_{i+1}^\infty \end{cases}$$
$$P[\phi(\overline{\pi_k}) = t] \stackrel{\text{def}}{=} \frac{1}{\#\{\pi_k \to_R t'\}} \text{ if } \pi_k \neq t_i^\infty \wedge \pi_k \to t.$$
$$P[\phi(\overline{\pi_k}) = t] \stackrel{\text{def}}{=} 0 \text{ if } \neg(\pi_k \to t)$$

The strategy ϕ is RR-fair, because it is fair and only the last state of the prefix of a derivation is used to compute the probability mesure used to select the next state of the current derivation. This kind of process is usually called memoryless or Markovian. Now, we show that there exists an infinite length derivation $(\pi_i)_{i \in \mathbb{N}}$ satifying $P[\exists n \; \pi_n = \bot] < 1$. Consider π the derivation under the aforementioned RR-strategy ϕ, with $\pi_0 = t_0^\infty$. Using the fact that the following event $\exists n, \pi_n = \bot$ complementary is $\forall n, \pi_n \neq \bot$, we deduce that:

$$P[\exists n, \pi_n = \bot] = 1 - P[\forall n, \pi_n \neq \bot].$$

Therefore, proving that $P[\exists n, \pi_n = \bot] < 1$ is logically equivalent to prove that $P[\forall n, \pi_n \neq \bot] > 0$. The event $\forall n \; \pi_n = t_n^\infty$ is a subset of the set of events such that $\forall n \; \pi_n \neq \bot$, henceforth :

$$P[\forall n, \pi_n = t_n^\infty] \leq P[\forall n, \pi_n \neq \bot],$$

with

$$P[\pi_n = t_n^\infty] = \prod_{i=1}^{n} P[\pi_n = t_n^\infty | \pi_{n-1} = t_{n-1}^\infty \wedge \ldots \wedge \pi_0 = t_0^\infty] \quad (4)$$
$$= \prod_{i=1}^{n} P[\pi_n = t_n^\infty | \pi_{n-1} = t_{n-1}^\infty] \quad (5)$$
$$= \prod_{i=1}^{n} (1 - e^{-i}).$$

Equation 5 is deduced from equation 4 using the fact that by definition ϕ is memoryless. To complete the proof, we have to show that:

$$\lim_{n\to\infty} P[\forall i \in \{1, \ldots, n\}\ \pi_i = t_i^\infty],$$

exists and is positive [1]. To show this, we use the conveniant properties of the classical Napier's Logarithm:

$$\ln P[\forall i \in \{1, \ldots, n\}\ \pi_i = t_i^\infty] = \ln(\prod_{i=1}^{n} (1 - e^{-i}))$$
$$= \sum_{i=1}^{i=n} \ln(1 - e^{-i})$$

Let be $I_n \stackrel{\text{def}}{=} \sum_{i=1}^{i=n} \ln(1 - e^{-i})$, using Taylor's series we get:

$$I_n = \sum_{i=1}^{i=n} \sum_{k \geq 1} -\frac{e^{-ik}}{k}$$
$$= -\sum_{k\geq 1} \frac{1}{k} \sum_{i=1}^{n} e^{-ik}$$
$$= -\sum_{k\geq 1} \frac{1}{k} e^{-k} \frac{1 - e^{-kn}}{1 - e^{-k}}$$

Therefore

$$\lim_{n\to\infty} I_n = -\sum_{k\geq 1} \frac{1}{k} e^{-k} \times \frac{1}{1 - e^{-k}}$$
$$\geq -\frac{1}{1 - e^{-1}} \times \sum_{k\geq 1} \frac{1}{k} e^{-k}$$
$$\geq \frac{1}{1 - e^{-1}} \times \ln(1 - e^{-1}),$$

[1] Non zero

hence,

$$P[\forall i \in \{1,\dots,n\}\ \pi_i = t_i^\infty] \geq (1-e^{-1})^{\frac{1}{1-e^{-1}}} > 0$$

therefore

$$P[\exists n, \pi_n = \bot] < 1,$$

which concludes the proof of the converse part, as well as the proof of theorem 3.

Theorem 4. *Let R be a fair terminating TRS. Then R is positively almost surely terminating under all RR-fair strategies.*

Informally, this theorem states that for every fairly terminating TRS R, firing the rewriting rules thanks to any RR-strategy will lead to a terminal term whithin a bounded mean number of rewrite steps.

Before starting the proof itself, we have to define some notations and random variables that will ease the proof readability. Consider ϕ a RR-strategy, and $\overline{\pi_k}$ a prefix of the derivation π, computed under strategy ϕ.

We define the stopping time $\tau_{\overline{\pi_k},n}$ as the random varible:

$$\tau_{\overline{\pi_k},n} \stackrel{\mathtt{def}}{=} min\{j \geq k : \pi_j \in pre^n\}$$

Proof. $\Rightarrow$

Let R be a fairly terminant TRS, ϕ a RR-fair strategy and $t \in T(\Sigma, V)$. We know thanks to theorem 1 that R is weakly terminating and that $\|t\| < \infty$. This implies that there exists $M \in \mathbb{N}$ such that for every $t \in T(\Sigma, V)$ $t \in pre^n$ with $n \in \{0, \dots, M\}$.

We show by induction that the following property $h(n)$ holds for all $n \in 1, \dots M$:

$$h(n) \stackrel{\mathtt{def}}{=} \begin{cases} \exists\ T_{n,n-1} < \infty \\ \forall\ \overline{\pi_k} \text{ with } \pi_k \in pre^n \\ E[\tau_{\overline{\pi_k},n-1}] \leq T_{n,n-1} \end{cases}$$

We first show that $h(N)$ holds.

Recall that because ϕ is RR-fair, we have that for any $n \in \{1, \dots, M\}$ $Pmin_\phi(n, n-1) > 0$. We now consider $\overline{\pi_k}$ a prefix of a derivation whose last term π_k belongs to pre^M. We compute here a finite upper bound to $E[\tau_{\overline{\pi_k},M-1}]$.

$$
\begin{aligned}
E[\tau_{\overline{\pi_k},M-1}] &= \sum_{i\geq 1} iP[\tau_{\overline{\pi_k},M-1} = i] \\
&= \sum_{i\geq 1} iP[\forall j < i \ \pi_{k+j} \in pre^M \wedge \pi_i \in pre^{M-1}] \\
&\leq \sum_{i\geq 1} i \prod_{k=1}^{i-1} (1 - Pmin_\phi(M, M-1)) \qquad (6) \\
&\leq \sum_{i\geq 1} i \times (1 - Pmin_\phi(M, M-1))^{i-1} \\
&\leq -\ln (Pmin_\phi(M, M-1)) \qquad (7)
\end{aligned}
$$

Equation 6 is deduced from the previous line by not considering the probability that $\pi_i \in pre^{M-1}$ and using the upper bound of

$$P[\forall j < i \ \pi_{k+j} \in pre^M \wedge \pi_i \in pre^{M-1}]$$

we provided in theorem 3 proof. In the same way, equation 7 is deduced from the line above using classical Taylor's series. One may remark that the real number $-\ln (Pmin_\phi(M, M-1))$ is a positive and finite value, as far as

$$0 < Pmin_\phi(M, M-1) \leq 1.$$

We now define

$$T_{M,M-1} \stackrel{\text{def}}{=} -\ln (Pmin_\phi(M, M-1)),$$

and this concludes the proof of $h(M)$.

Let us now prove that if for all $l \in \{n, \ldots, M\}$ property $h(l)$ holds, then $h(n-1)$ holds.

To do so, let us first show, that for any derivation π under ϕ with $\pi_k \in pre^{n+j}$, with $n + j \leq M$, we have the following equation :

$$E[\tau_{\overline{\pi_k},n}] \leq \sum_{i=0}^{j-1} T_{n+i+1,n+i} \qquad (8)$$

To prove equation 8, let's just write:

$$
\begin{aligned}
E[\tau_{\overline{\pi_k},n}] &= E[\tau_{\overline{\pi_k},n+m-1} + \tau_{\overline{\pi_{\tau_{\overline{\pi_k},n+j-1}}},n+j-2} + \ldots + \tau_{\overline{\pi_{\cdot_{\cdot_{\cdot_{\tau_{\overline{\pi_k},n+j-1}}}}}},n}] \\
&\leq E[\tau_{\overline{\pi_k},n+j-1}] + \ldots + E[\tau_{\overline{\pi_{\cdot_{\cdot_{\cdot_{\tau_{\overline{\pi_k},n+j-1}}}}}},n}] \\
&\leq \sum_{i=0}^{j-1} T_{n+i+1,n+i}
\end{aligned}
$$

We can now start the proof of induction step: To do this, we show that there exists $T_{n,n-1} < \infty$ s.t. $E[\tau_{\overline{\pi_k},n-1}] \leq T_{n,n-1}$. Let us recall that we showed in theorem 3 proof that the only way for π to reach a term of pre^{n-1}, when all terms of $\overline{\pi_k}$ belongs to $P_{M,n}$ is to reach pre^n first. Once $\pi_i \in pre^n$, for some i, π_{i+1} may belongs to pre^{n-1} with probability greater or equal than $Pmin_\phi(n, n-1)$, or may belong to $P_{M,n}$ with the complementary probability, at most $1-Pmin_\phi(n, n-1)$. If $\pi_i \in P_{M,n+1}$, then $P[\pi_{i+1} \in pre^{n-1}] = 0$.

Let us compute an upper bound of $E[\tau_{\overline{\pi_k},n-1}] \leq T_{n,n-1}$, by using the equality

$$E[\tau_{\overline{\pi_k},n-1}] = E[E[\tau_{\overline{\pi_k},n-1}|(\tau_{i,n})_{i\in\mathbb{N}}]] \tag{9}$$

Equation 9 holds because $\forall i \in \mathbb{N}$ the σ field $\sigma(\bigcup_{k=0}^{i} \tau_{k,n})$ contains the trivial σ-field $\{\emptyset, \Omega\}$.

We compute $E[\tau_{\overline{\pi_k},n-1}|(\tau_{i,n})_{i\in\mathbb{N}}]$

$$\begin{aligned} E[\tau_{\overline{\pi_k},n-1}|(\tau_{i,n})_{i\in\mathbb{N}}] = \quad & (\tau_{1,n}+1) \times P[\phi(\overline{\pi_{\tau_{1,n}}}) \in pre^{n-1}] \\ & +(\tau_{2,n}+1) \times P[\phi(\overline{\pi_{\tau_{2,n}}}) \in pre^{n-1}] \times P[\phi(\overline{\pi_{\tau_{1,n}}}) \notin pre^{n-1}] \\ & \vdots \\ & +(\tau_{k+1,n}+1) \times \quad P[\phi(\overline{\pi_{\tau_{k,n}}}) \in pre^{n-1}] \times \prod_{j=1}^{k-1} P[\phi(\overline{\pi_{\tau_{j,n}}}) \notin pre^{n-1}] \\ & \vdots \end{aligned}$$

In a compact form we have:

$$\begin{aligned} E[\tau_{\overline{t_k},n-1}|(\tau_{i,n})_{i\in\mathbb{N}}] = \quad & (\tau_{1,n}+1) \times P[\phi(\overline{\pi_{\tau_{1,n}}}) \in pre^{n-1}] \\ & + \sum_{k\geq 2} (\tau_{k,n}+1) \times \quad P[\phi(\overline{\pi_{\tau_{k,n}}}) \in pre^{n-1}] \times \prod_{j=1}^{k-1} P[\phi(\overline{\pi_{\tau_{j,n}}}) \notin pre^{n-1}] \end{aligned}$$

For computing an upper bound of $E[\tau_{\overline{\pi_k},n-1}]$, we just compute an upper bound of $E[E[\tau_{\overline{\pi_k},n-1}|(\tau_{i,n})_{i\in\mathbb{N}}]]$

$$\begin{aligned} E[\tau_{\overline{\pi_k},n-1}] &= E[E[\tau_{\overline{\pi_k},n-1}|(\tau_{i,n})_{i\in\mathbb{N}}]] \\ E[\tau_{\overline{\pi_k},n-1}] &= E\left[\begin{array}{c} (\tau_{1,n}+1) \times P[\phi(\overline{\pi_{\tau_{1,n}}}) \in pre^{n-1}] \\ + \sum_{k\geq 2} (\tau_{k+1,n}) \times P[\phi(\overline{\pi_{\tau_{k,n}}}) \in pre^{n-1}] \times \prod_{j=1}^{k-1} P[\phi(\overline{\pi_{\tau_{j,n}}}) \notin pre^{n-1}] \end{array}\right] \\ &\leq 1 + E[\tau_{1,n}] + \sum_{i\geq 2} (1 - Pmin_\phi(n, n-1))^i \times E[(\tau_{i,n}+1)] \\ &\leq 1 + T_{N,n} + \sum_{i\geq 2} T_{N,n} \times i \times (1 - Pmin_\phi(n, n-1))^i \\ &\leq 1 + T_{N,n} + \sum_{i\geq 2} T_{N,n} \times i \times (1 - Pmin_\phi(n, n-1))^{(i-1)} \\ &\leq 1 + T_{N,n}(1 - \ln(Pmin_\phi(n, n-1))). \end{aligned}$$

We proved indeed that: $T_{n,n-1}$ exists and

$$T_{n,n-1} = 1 + T_{N,n}(1 - \ln(Pmin_{\phi}(n, n-1))),$$

which concludes the proof.